

## Floor and Ceiling

We are given

$$(0a) \quad \langle \forall n : n \leq x : n \leq \lfloor x \rfloor \rangle$$

$$(1a) \quad \lfloor x \rfloor \leq x$$

$$(0b) \quad \langle \forall n : x \leq n : \lceil x \rceil \leq n \rangle$$

$$(1b) \quad x \leq \lceil x \rceil$$

We will start by proving

$$(2a) \quad n \leq \lfloor x \rfloor \equiv n \leq x$$

$$(2b) \quad \lceil x \rceil \leq n \equiv x \leq n$$

by mutual implication.

$$\begin{aligned} n \leq x &\Rightarrow n \leq \lfloor x \rfloor \\ = \{ (0a) \} \\ &\text{true} \end{aligned}$$

$$\begin{aligned} x \leq n &\Rightarrow \lceil x \rceil \leq n \\ = \{ (0b) \} \\ &\text{true} \end{aligned}$$

$$\begin{aligned} n \leq \lfloor x \rfloor \\ \Rightarrow \{ (1a); \text{transitivity} \} \\ n \leq x \end{aligned}$$

$$\begin{aligned} \lceil x \rceil \leq n \\ \Rightarrow \{ (1b); \text{transitivity} \} \\ x \leq n \end{aligned}$$

Next we observe

$$\begin{aligned} n \leq \lfloor x \rfloor &\equiv n \leq x \\ = \{ \text{predicate calculus} \} \\ \neg(n \leq \lfloor x \rfloor) &\equiv \neg(n \leq x) \\ = \{ \text{arithmetic} \} \\ \lfloor x \rfloor < n &\equiv x < n \end{aligned}$$

$$\begin{aligned} \lceil x \rceil \leq n &\equiv x \leq n \\ = \{ \text{predicate calculus} \} \\ \neg(\lceil x \rceil \leq n) &\equiv \neg(x \leq n) \\ = \{ \text{arithmetic} \} \\ n < \lceil x \rceil &\equiv n < x \end{aligned}$$

3(e) asks us to show

$$\lfloor x \rfloor = n \equiv x - 1 < n \leq x < n + 1$$

We observe

$$\begin{aligned} x - 1 < n \wedge n \leq x \wedge x < n + 1 \\ = \{ \text{arithmetic; idempotency} \} \\ n \leq x \wedge x < n + 1 \end{aligned}$$

Given (2a) and antisymmetry of  $\leq$  it remains to show

$$x < n + 1 \equiv n \geq \lfloor x \rfloor$$

. We calculate

$$\begin{aligned} x < n + 1 \\ = \{ x < n \equiv \lfloor x \rfloor < n \} \\ \lfloor x \rfloor < n + 1 \\ = \{ \text{integers} \} \\ \lfloor x \rfloor \leq n \end{aligned}$$

Finally we must show

$$\lfloor x \rfloor = n \equiv x \leq n < x + 1 \wedge n - 1 < x \leq n$$

Here we go:

$$\begin{aligned} & x \leq n \quad \wedge \quad n < x+1 \quad \wedge \quad n-1 < x \quad \wedge \quad x \leq n \\ = & \quad \{\text{arithmetic; idempotency}\} \\ & x \leq n \quad \wedge \quad n < x+1 \end{aligned}$$

Once again the antisymmetry of  $\leq$  and this time (2b) means we need only show

$$n < x+1 \quad \equiv \quad n \leq \lceil x \rceil$$

We observe

$$\begin{aligned} & n < x+1 \\ = & \quad \{\text{arithmetic}\} \\ & n-1 < x \\ = & \quad \{n < \lceil x \rceil \equiv n < x\} \\ & n-1 < \lceil x \rceil \\ = & \quad \{\text{arithmetic}\} \\ & n < \lceil x \rceil + 1 \\ = & \quad \{\text{arithmetic}\} \\ & n \leq \lceil x \rceil \end{aligned}$$

All done.

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*13 July 2006*