

Given

$$b^0 = 1, \quad b^n = b^{n-1} \text{ for } n > 0$$

we are asked to prove the laws of exponents:

$$(0) \quad b^{x+y} = b^x * b^y$$

$$(1) \quad (b^x)^y = b^{x*y}$$

where x and y are integers.

The definition of b^n suggests a proof by induction. Let us begin with (0).

$$\underline{b^{x+y} = b^x * b^y}$$

We are tasked with proving $\langle \forall x, y :: b^{x+y} = b^x * b^y \rangle$.

We observe

$$\begin{aligned} & \langle \forall x, y :: b^{x+y} = b^x * b^y \rangle \\ = & \quad \{\text{nesting}\} \\ & \langle \forall y :: \langle \forall x :: b^{x+y} = b^x * b^y \rangle \rangle \end{aligned}$$

We thus prove $\langle \forall y :: P.y \rangle$ where $P.y = \langle \forall x :: b^{x+y} = b^x * b^y \rangle$

Base case $b^{x+0} = b^x * b^0$

$$\begin{aligned} & b^{x+0} = b^x * b^0 \\ = & \quad \{\text{arithmetic; definition of } b^n\} \\ & b^x = b^x * 1 \\ = & \quad \{\text{identity of } *; \text{ reflexivity}\} \\ & \text{true} \end{aligned}$$

Inductive step $b^{x+(y+1)} = b^x * b^{y+1}$

$$\begin{aligned} & b^{x+(y+1)} \\ = & \quad \{\text{properties of } +\} \\ & b^{(x+1)+y} \\ = & \quad \{\text{inductive hypothesis}\} \\ & b^{x+1} * b^y \\ = & \quad \{\text{definition of } b^n\} \\ & b^x * b^1 * b^y \\ = & \quad \{\text{properties of } *\} \\ & b^x * b^y * b^1 \\ = & \quad \{\text{definition of } b^n\} \end{aligned}$$

$$b^x * b^{y+1}$$

$$\underline{(b^x)^y = b^{x*y}}$$

Base case $(b^x)^0 = b^{x*0}$

$$(b^x)^0 = b^{x*0}$$

= {definition of b^n , twice; arithmetic}

$$1 = 1$$

= {reflexivity}

true

Base case $(b^x)^1 = b^{x*1}$

$$(b^x)^1 = b^{x*1}$$

= {definition of b^n ; arithmetic}

$$b^x = b^x$$

= {reflexivity}

true

Inductive step $(b^x)^{y+1} = b^{x*(y+1)}$

$$(b^x)^{y+1}$$

= {definition of b^n }

$$(b^x)^y * (b^x)^1$$

= {induction hypothesis}

$$b^{x*y} * b^{x*1}$$

= {(0)}

$$b^{(x*y)+(x*1)}$$

= {factor out x }

$$b^{x*(y+1)}$$

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