

A problem courtesy of Shakespeare

This note is about a powerful tool for solving problems, viz. calculation. Consider the following problem from Shakespeare's *The Merchant of Venice*:

Portia has a gold casket and a silver casket and has placed a picture of herself in one of them. On the caskets she has written the following inscriptions:

Gold: The portrait is not here.

Silver: Exactly one of these inscriptions is true.

Portia explains to her suitor that each inscription may be true or false, but that she has placed her portrait in one of the caskets in a manner that is consistent with this truth or falsity of the inscriptions. If he can choose the casket with her portrait, she will marry him. The problem for the suitor is to use the inscriptions (although they may be either true or false) to determine which casket contains her portrait.

How can we solve this problem?

* *
 *
 *

Equivalence

We already know how to calculate with numbers. For example we can simplify $((1 + 2) - 3 * 4)/9$ as follows:

$$\begin{aligned} & ((1 + 2) - 3 * 4)/9 \\ = & \{ 1 + 2 = 3 \} \\ & (3 - 3 * 4)/9 \\ = & \{ 3 * 4 = 12 \} \\ & (3 - 12)/9 \\ = & \{ 3 - 12 = -9 \} \\ & -9/9 \\ = & \{ -9/9 = -1 \} \\ & -1 \end{aligned}$$

To solve puzzles like Portia's we need to learn how to calculate not just with numbers but with *propositions*.

* *

 *

Suppose we have two propositions X and Y . To say ‘proposition X is equivalent to proposition Y ’ we write $X \equiv Y$. Notice that $X \equiv Y$ is also a proposition.

A proposition can be either true or false. If a proposition X is true we write $X \equiv \top$. Now saying ‘it is true that is raining’ is the same as just saying ‘it is raining.’ We use this rule in calculations like so

$$\begin{aligned} & X \equiv \top \\ \equiv & \quad \{ \text{identity} \} \\ & X \end{aligned}$$

* *

 *

A proposition is equivalent to itself. Hence the proposition ‘ X is equivalent to X ’ is true. This rule is called *reflexivity*. How do we use it in calculations? Like so:

$$\begin{aligned} & X \equiv X \\ = & \quad \{ \text{reflexivity} \} \\ & \top \end{aligned}$$

The proposition “ X is equivalent to Y ” is equivalent “ Y is equivalent to X ” is true. When using this rule in calculations we use the catchword ‘symmetry’:

$$\begin{aligned} & X \equiv Y \\ = & \quad \{ \text{symmetry} \} \\ & Y \equiv X \end{aligned}$$

Finally, If a proposition X is false we write $X \equiv \perp$.

* *

 *

Inequivalence

We have seen how to deal with equivalent propositions. But what about propositions that are not equivalent? To express that proposition X differs from proposition Y we write $X \not\equiv Y$. Like \equiv , $\not\equiv$ is symmetric:

$$\begin{aligned} & X \not\equiv Y \\ = & \quad \{ \text{symmetry} \} \\ & Y \not\equiv X \end{aligned}$$

In addition $\not\equiv$ and \equiv are ‘mutually associative’:

$$\begin{aligned} & (X \not\equiv Y) \equiv Z \\ \equiv & \quad \{ \text{mutual associativity} \} \\ & X \not\equiv (Y \equiv Z) \end{aligned}$$

* *

 *

Let us return to the problem posed by Portia. We start by naming some propositions:

G : The portrait is in the gold casket.

S : The portrait is in the silver casket.

IG : The portrait is not in the gold casket.

IS : Exactly one of IG and IS is true.

We observe

- The inscription on the gold casket says that the portrait is not inside. This differs from proposition G which we capture with the formula $IG \not\equiv G$.
- Next, the fact that the portrait can only be in one casket is expressed by the formula $G \not\equiv S$.
- The inscription on the silver casket can be expressed by $IS \equiv (IG \not\equiv IS)$

Alright, let us try to calculate. We will start with the most complicated formula as it offers us the greatest manipulation opportunities.

$$\begin{aligned}
& IS \equiv (IG \not\equiv IS) \\
= & \quad \{ \text{We see an opportunity to obtain } IS \equiv IS \text{ which would allow us to} \\
& \quad \text{vastly simplify the formula. In preparation for that we exploit the symmetry of } \equiv \} \\
& (IG \not\equiv IS) \equiv IS \\
= & \quad \{ \text{mutual associativity} \} \\
& IG \not\equiv (IS \equiv IS) \\
= & \quad \{ \text{reflexivity} \} \\
& IG \not\equiv \top
\end{aligned}$$

We have calculated that the inscription on the gold casket is not true i.e. Portia's portrait *is* in the gold casket.

2007.11.29

Eric Emmanuel Macaulay