

The maximum pair difference

Given integer array A of length N , $1 \leq N$, we are asked to compute the maximum pair difference for A . Formally, for natural numbers p and q and integer r our program establishes

$$R: \quad r = \langle \uparrow p, q: 0 \leq p < q < N: A.p - A.q \rangle$$

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We investigate our standard invariant, P , which is the conjunction of

$$Pr: \quad r = \langle \uparrow p, q: 0 \leq p < q < n: A.p - A.q \rangle$$

$$Pn: \quad 0 \leq n \leq N$$

which we initially establish with $n, r := 0, -\infty$. We are heading for a program of the form

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n, r, := 0, -∞
; do n ≠ N →
    “Increase n by 1 under invariance of P ”
od

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The proofs of termination and maintenance of Pn under $n := n + 1$ are standard and trivial and we thus omit them.

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We observe

$$\begin{aligned}
 & \llbracket \text{Context: } P \wedge n \neq N \\
 & \quad \langle \uparrow p, q: 0 \leq p < q < n+1: A.p - A.q \rangle \\
 & = \{ \text{split off } q = n \text{ requires } n < N \text{ which follows from } Pn \wedge n \neq N \} \\
 & \quad \langle \uparrow p, q: 0 \leq p < q < n: A.p - A.q \rangle \uparrow \langle \uparrow p: 0 \leq p < n: A.p - A.n \rangle \\
 & = \{ Pr \} \\
 & \quad r \uparrow \langle \uparrow p: 0 \leq p < n: A.p - A.n \rangle \\
 & = \{ + \text{ over } \uparrow \} \\
 & \quad r \uparrow (\langle \uparrow p: 0 \leq p < n: A.p \rangle - A.n) \\
 & = \{ Ps \} \\
 & \quad r \uparrow (s - A.n) \\
 & \rrbracket
 \end{aligned}$$

where we strengthen P with

$$Ps : \quad s = \langle \uparrow p : 0 \leq p < n : A.p \rangle$$

We initially establish Ps with $n, s := 0, -\infty$.

At this point we have

$$\begin{aligned} &[[\textit{Context}: P \wedge n \neq N \\ &\quad \{ Pr \wedge Ps \} \\ &\quad r := r \uparrow (s - A.n) \\ &\quad \{ (n := n + 1).Pr \} \\ &]] \end{aligned}$$

giving us a program of the form

$$\begin{aligned} &n, r, s := 0, -\infty, -\infty \\ &; \mathbf{do} \quad n \neq N \quad \rightarrow \\ &\quad \text{"Maintain } Ps \text{ under } n, r := n + 1, r \uparrow (s - A.n) \text{"} \\ &\mathbf{od} \end{aligned}$$

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We turn our attention to the maintenance of Ps . We observe

$$\begin{aligned} &[[\textit{Context}: P \wedge n \neq N \\ &\quad \langle \uparrow p : 0 \leq p < n + 1 : A.p \rangle \\ &= \quad \{ \text{split off } p = n, \text{ requires } 0 \leq n \text{ which follows from } Pn \} \\ &\quad \langle \uparrow p : 0 \leq p < n : A.p \rangle \uparrow A.n \\ &= \quad \{ Ps \} \\ &\quad s \uparrow A.n \\ &]] \end{aligned}$$

We now have

$$\begin{aligned} &[[\textit{Context}: P \wedge n \neq N \\ &\quad \{ Ps \} \quad s := s \uparrow A.n \quad \{ (n := n + 1).Ps \} \\ &]] \end{aligned}$$

hence our solution

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 $n, r, s := 0, -\infty, -\infty$   
; do  $n \neq N \rightarrow$   
     $n, r, s := n + 1, r \uparrow (s - A.n), s \uparrow A.n$   
od
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